ID Number: Key

Test 1A

1. (16 points) Use the limit definition of the derivative to calculate $f'(x)$ if

$$f(x) = \frac{3}{x^2}.$$

Make sure you use correct notation.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \left(\frac{3}{(x+h)^2} - \frac{3}{x^2} \right) \frac{1}{h}$$

$$= \lim_{h \to 0} \left(\frac{3x^2 - 3(x+h)^2}{(x+h)^2 x^2} \right) \frac{1}{h}$$

$$= \lim_{h \to 0} \left(\frac{\cancel{3x^2} - (\cancel{3x^2} + 6xh + 3h^2)}{(x+h)^2 x^2} \right) \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-6x - 3h)}{(x+h)^2 x^2} \cdot \frac{1}{\cancel{h}}$$

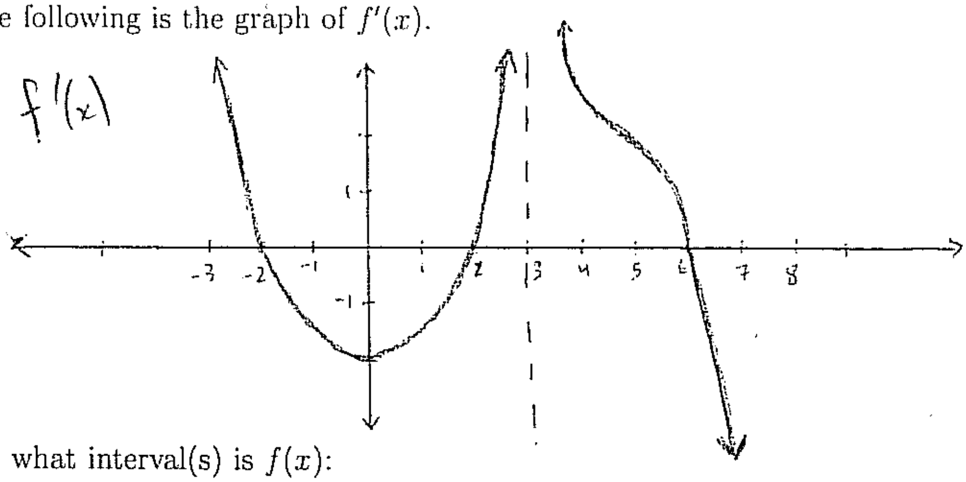
$$= \lim_{h \to 0} \frac{-6x \cancel{- 3h}}{(x + \cancel{h})^2 x^2}$$

$$= \frac{-6\cancel{x}}{x^{\cancel{4}3}}$$

$$= \boxed{\frac{-6}{x^3}}$$

✓✓ for notation.

2. (9 points) For this question, pay close attention to which one is f and which one is f'. The following is the graph of $f'(x)$.



On what interval(s) is $f(x)$:

(a) Increasing $(-\infty, -2)$, $(2, 3)$, $(3, 6)$ ✓✓✓

(b) Decreasing $(-2, 2)$ and $(6, \infty)$ ✓✓✓

(c) Concave down $(-\infty, 0)$, $(3, \infty)$ ✓✓✓

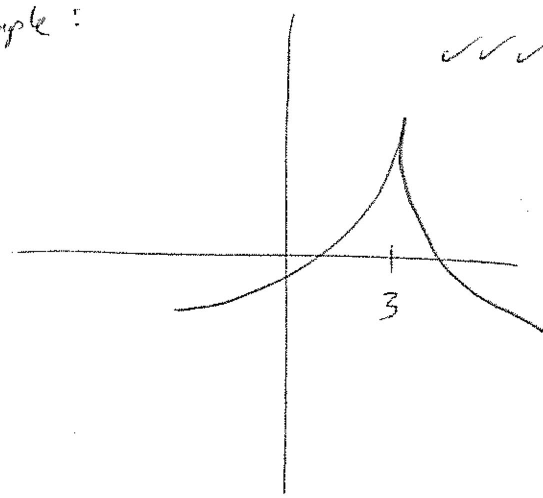
3. (6 points) Is the statement below true or false? If true, briefly explain why. If false, give a counterexample or explain your reasoning.

(a) " If $\lim_{x \to 3} f(x) = f(3)$, then $f'(3)$ exists. "

✓✓

False. Counterexample:

Can have continuity but not be differentiable.



✓✓✓

4. (16 points) Use the derivative laws (i.e. not the definition) to calculate the equation of the tangent line to

$$f(x) = \frac{x^3 + 2\sqrt{x^3} - 7}{x}$$

at $x = 1$.

$$f(x) = \frac{x^3}{x} + \frac{2x^{3/2}}{x} - \frac{7}{x}$$

$$f(x) = x^2 + 2x^{1/2} - 7x^{-1}$$

$$f'(x) = \frac{d}{dx}x^2 + 2\frac{d}{dx}(x^{1/2}) - 7\frac{d}{dx}(x^{-1})$$

$$= 2x + 2\cdot\frac{1}{2}x^{-1/2} - 7(-1x^{-2})$$

$$= 2x + x^{-1/2} + 7x^{-2}$$

$$f'(1) = 2 + 1 + 7 = 10$$

$$f(1) = 1 + 2 - 7 = -4 = y_1$$

$$y - y_1 = f'(1)(x - x_1)$$

$$y - (-4) = 10(x - 1) \Rightarrow \boxed{\begin{array}{l} y + 4 = 10x - 10 \\ y = 10x - 14 \end{array}}$$

$$\cancel{y + 4 = 6.5x - 6.5}$$

$$\cancel{y = 6.5x - 10.5}$$

5. (15 points) Find all horizontal and vertical asymptotes of

$$\frac{3x^2 - 3x - 60}{x^2 - 16}.$$

Explain your answer, and make sure to use correct notation.

horizontal asymptotes are found by looking at

$$\lim_{x \to \pm\infty} \frac{3x^2-3x-60}{x^2-16} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

(7)

$$= \lim_{x\to\pm\infty} \frac{3 - \cancel{\frac{3}{x}} - \cancel{\frac{60}{x^2}}}{1 - \cancel{\frac{16}{x^2}}}$$

$$= \frac{3}{1} = 3$$ so $y=3$ is the only horizontal asymptote

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vertical asymptotes are where $\lim_{x\to p^\pm} f(x) = \pm\infty$,

which is where denominator is zero, usually. $x^2-16 = (x+4)(x-4)$

(8)

so $\lim_{x\to 4} \frac{3x^2-3x-60}{x^2-16} = \lim_{x\to 4} \frac{3(x^2-x-20)}{(x+4)(x-4)} = \lim_{x\to 4} \frac{3(x-5)\cancel{(x+4)}}{\cancel{(x+4)}(x-4)} = \text{dne}$

$\lim_{x\to 4^-} \frac{3(x-5)}{(x-4)} = \infty$

$\Rightarrow$ $\boxed{x=4 \text{ is a vertical asymptote}}$

but, $\lim_{x\to -4} \frac{3x^2-3x-60}{x^2-16} = \lim_{x\to -4} \frac{3(x-5)}{x-4}$

$$= \frac{3(-9)}{-8} = \frac{27}{8}$$

$x=-4$ is not a v.a.

6. (16 points) Find the value of c which will make the following function continuous at $x = 9$. Make sure to use correct notation and explain your reasoning:

$$f(x) = \begin{cases} \left[\dfrac{\sqrt{x}-3}{x-9}\right] & \text{if } x > 9 \\ 2c & \text{if } x \leq 9 \end{cases}$$

Need $\lim_{x\to 9} f(x) = f(9)$

$f(9) = 2c$ and $\lim_{x\to 9^-} f(x) = 2c$

$$\lim_{x\to 9^+} f(x) = \lim_{x\to 9^+} \frac{\sqrt{x}-3}{x-9} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3}$$

$$= \lim_{x\to 9^+} \frac{\cancel{x-9}}{\cancel{(x-9)}(\sqrt{x}+3)}$$

$$= \lim_{x\to 9^+} \frac{1}{\sqrt{x}+3}$$

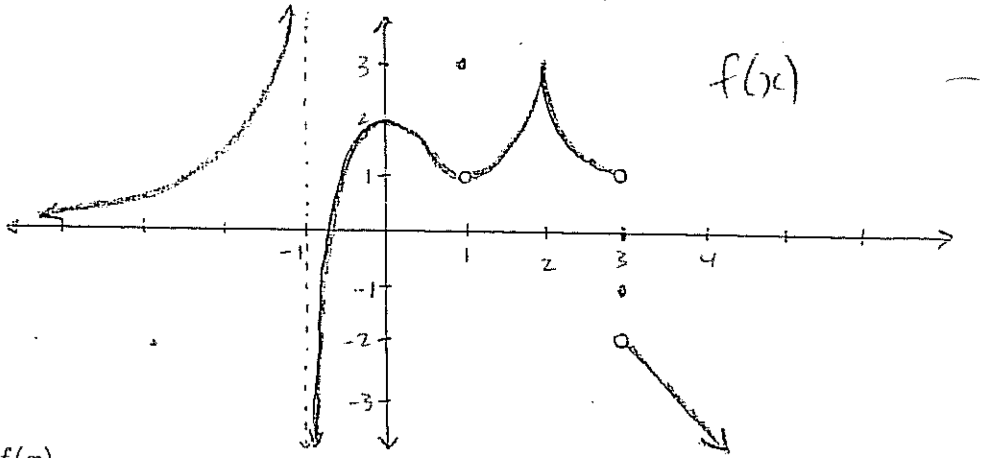
$$= \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

Need c such that

$$2c = \frac{1}{6}$$

$$\Rightarrow \boxed{c = \frac{1}{12}}$$

7. (10 points) For the given graph, find the value for the expressions below. If the expression does not exist, give a (brief) explanation of why not.



(i) $\lim_{x\to(-1)^-} f(x)$ does not exist, $= \infty$

✓✓

(ii) $\lim_{x\to 2^+} f(x) = 3$

✓✓

(iii) $\lim_{x\to(-\infty)} f(x) = 0$

✓✓

(iv) $\lim_{x\to 3} f(x)$ does not exist.

✓✓

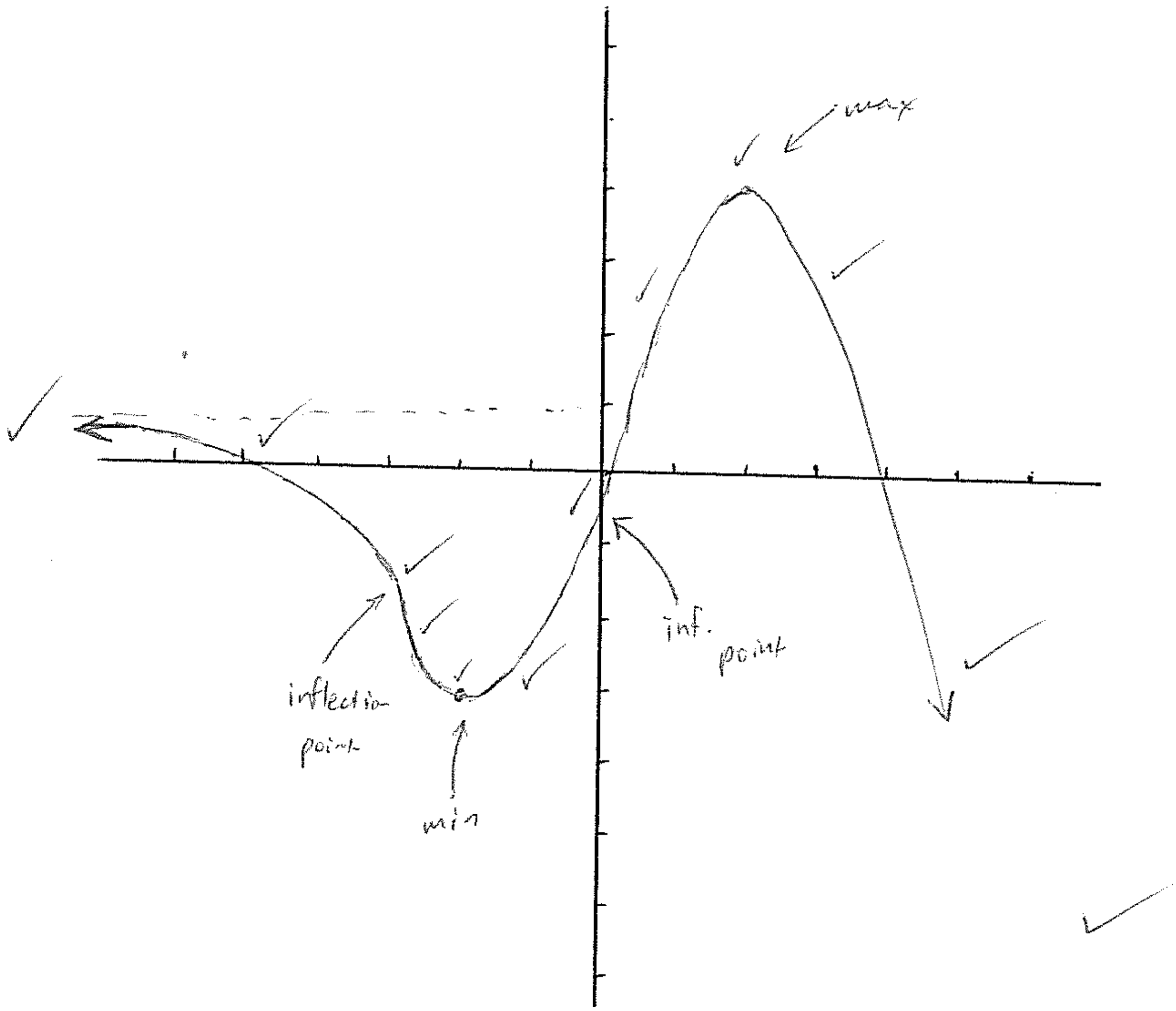
$\lim_{x\to 3^-} f(x) = 1$

$\lim_{x\to 3^+} f(x) = -2$

✓✓ (v) $f'(1)$ does not exist because there is a discontinuity.

8. (12 points) Sketch the graph of a <u>continuous</u> function $f(x)$ that satisfies the following conditions. Label all local maximums, local minimums, and inflection points on your graph:

- $f(-2) = -3$ and $\lim_{x \to 2} f(x) = 4$
- $f'(-2) = 0$ and $f'(2) = 0$
- $f'(x) < 0$ on the intervals $(-\infty, -2)$ and $(2, \infty)$; $f'(x) > 0$ on the interval $(-2, 2)$.
- $f''(x) < 0$ on the intervals $(-\infty, -3)$ and $(0, \infty)$; $f''(x) > 0$ on the interval $(-3, 0)$
- $\lim_{x \to -\infty} f(x) = 1$



Extra Credit(2 points) Use the derivative rules (<u>not</u> the limit definition) to find $f'(x)$ if $f(x) = e^7$

$f'(x) = 0$ b/c e^7 is a constant